AP STATISTICS
TOPIC 2: COUNTING

PAUL L. BAILEY

1. Foundational Principles

Principle 1. (Multiplication Principle)
If a task can be performed in two independent stages, and the first stage can performed in n_1 ways, and the second stage can be performed in n_2 ways, then the entire task can be performed in $n_1 n_2$ ways.

Example 1. If we have five shirts and three pairs of pants, and we pick a shirt and then a pair of pants to wear, there are $5 \cdot 3 = 15$ different ensembles we can wear

Principle 2. (Addition Principle)
If a task can be split into two mutually exclusive subtasks, and the first subtask can be performed in n_1 ways, and the second subtask can be performed in n_2 ways, then the entire task can be performed in $n_1 + n_2$ ways.

Example 2. If we have 15 shirt-and-pant ensembles to choose from, or instead we could wear one of 7 togas, we have $15 + 7 = 22$ different outfits we can wear.

2. Arranging

Suppose we have a set of n objects. We wish to count the number of ways there are to arrange (or rearrange) these objects in a list. We image a sequence of n slots, and realize that we wish to place one of the objects in each slot. We have n choices for the first slot, $n-1$ choices for the second slot, so the number of ways to begin our arrangement with its first two elements is $n(n-1)$, by the multiplication principle. Continuing, we have $n-2$ ways to choose the third object, and so forth, continuing to multiply until we see that there are

$$n! = n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1$$

ways to arrange n objects. That is, there are $n!$ different lists of distinct objects we can make out of a set of n objects.

Date: August 27, 2024.

3. Sampling

Suppose we have a set of n objects, and we wish to select k objects from this set. Our choice of objects is called a sample, and the act of choosing them is called sampling. There are a few different methods of sampling of interest.

- *Ordered sampling*: We keep track of the order in which the objects are selected. Thus, if we select the same objects but in a different order, this is consider a different sample.
- *Unordered sampling*: The order in which objects are select does not matter. The same set of objects, selected in a different order, would be considered the same sample.
- *With replacement*: We could select an object, note which one was selected, and then return the object so that it could potentially be selected again. That is, we could allow repetitions.
- *Without replacement*: Once we select an object, we may not select it again.

This gives four different combinations of these two ideas. Each one produces a different number of distinct samples. We look at each.

We wish to count the number of ways to select k things from a set of n things. Note that when we are counting with replacement, it is possible that $n < k$, but if we are counting without replacement, it is necessary that $k \leq n$.

3.1. Ordered Sampling with Replacement.

To count the number of ways to select k things from a set of n things with order and replacement, we repetitively apply the multiplication principle. We imagine filling k slots with objects from the set. There are n choices for slot 1, and n choices for slot 2, so there are n^2 choices for the first two. There are n choices for the third, and n choices for the forth, up to n choices for the k; we continue to multiply by n and obtain

$$\text{Ordered with Replacement Selection gives } n^k \text{ samples.}$$

Note that in this case, it is possible that $n < k$.

3.2. Ordered Sampling without Replacement.

To count the number of ways to select k things from a set of n things with order but without replacement, we first select one for the first slot, and have n choices; now, however, we have one less choice for the second slot, so we multiply by $n-1$. There are $n-2$ choices for the third slot, and so forth. We do this k times, multiplying as we go, and see that

$$n(n-1)(n-2)\cdots(n-k+1) = \frac{n!}{(n-k)!},$$

to obtain

$$\text{Ordered without Replacement Selection gives } \frac{n!}{(n-k)!} \text{ samples.}$$

An ordered list of distinct objects is called a *permutation*. The number of permutations of k objects from a set of n objects is denoted $P(n,k)$.

3.3. Unordered Sampling without Replacement.

Here, we use the last result. Consider an unordered sample of size k, chosen without duplicates from a set of n objects. There are $k!$ ways to arrangement this sample into an ordered list, which implies that in our above computation of order sample without replacement, each such set was counted $k!$ times. Thus, we divide the number of ordered samples without replacement by $k!$, to obtain

$$\text{Unordered without Replacement Selection gives } \frac{n!}{k!(n-k)!} \text{ samples.}$$

An unordered set of distinct objects is called a *combination*. The number of combinations of k objects from a set of n objects is denoted $C(n,k)$, or $\binom{n}{k}$. The latter notation is normally read n *choose* k. Note that

$$C(n,k) = \binom{n}{k} \quad \text{and} \quad P(n,k) = \binom{n}{k}k!.$$

3.4. **Unordered Sampling with Replacement.** This is the most difficult of the four types of sampling to grasp; the most common explanation requires that we rethink our counting paradigm.

Instead of imagining placing objects in slots, we imagine the object being in one of several adjacent bins.

Now imagine walls that divide the bins, separating the balls into the different bins. This requires $n-1$ walls. For example, suppose $n = 4$ and $k = 7$. We can produce diagrams for how the balls are thrown into the bins; for example:

```
*|**|*|***       1, 2, 1, 3
**|***||**       2, 3, 2
||*******        0, 0, 7
```

We see that if we were to list all possible ways of doing this, we could image a sequence of slots, and into each slot we would place either a ball or a wall. There are k balls and $n-1$ walls, so this would require $n+k-1$ slots. A given sample consists of selected where to place the walls (or the balls) into these $n+k-1$ slots. There are $\binom{n+k-1}{k} = \binom{n+k-1}{n-1}$ ways to do this. We obtain

$$\text{Unordered with Replacement Selection gives } \binom{n+k-1}{k} \text{ samples.}$$

3.5. **Summary.** Our four counting techniques produce four formulae, which we summarize here. If we select k objects from a set of n objects, we recapitulate the number of ways to do this with each counting technique.

	Ordered	Unordered
With Replacement	n^k	$\binom{n+k-1}{k}$
Without Replacement	$\frac{n!}{(n-k)!}$	$\binom{n}{k}$

4. Problems

Problem 1. A coin is flipped ten times. Determine the sample space, and find the probability of each event. Count the number of possible outcomes.

Problem 2. A bin contains 60 balls. Ten balls are drawn at random. Count the number of possible outcomes.

Problem 3. Five cards are dealt from a shuffled deck and placed in a line. Count the number of possible outcomes.

Problem 4. Three dice of different colors are rolled. Count the number of possible outcomes.

Problem 5. Three indistinguishable dice are rolled. Count the number of detectable outcomes.

Problem 6. How many polynomials of degree 5 have only integer zeros in the range from 0 to 10 (no other complex zeros)?

Department of Mathematics, Paragon Science Academy
Email address: pbailey@sonoranschools.org